**Design Techniques For New Engineers**
*Units and Pseudo-Units*
*by Dennis L Feucht*

Physical quantities are represented mathematically by numbers indicating their *amount*. Depending on whether the quantity is discrete or continuous, integers, rational numbers, or real numbers will be used. Quantities such as the complex frequency, *s*, require complex numbers. All of these numbers, however, are incomplete in themselves in that they do not adequately represent *physical* quantities. The additional connection between numbers and the physical world is achieved through the use of units.

A *unit* is the amount of a quantity that is one of it, the "unity" amount: hence the name for unit. A unit is something that exists in the physical world, not in mathematics as such, and is brought into math as a symbol that is multiplied to the number representing the amount of the quantity. For instance, through physical description, an ampere of current is a certain amount of the familiar quantity, with a standard abbreviation of A. If the amount of current in amperes is a number, *n*, then *n* A is a complete mathematical description of the *value* of the quantity.



## Units And Computer Math

Computer math programs have necessitated the refinement of mathematical notation in that a more rigorous way to specify the quantity is $n \cdot \mathrm{A}$. This is *n* times the amount of one ampere, which has an unambiguous physical value. As anyone who has used them knows, math programs will also not tolerate implicit designation of multiplication. The expression, *ab*, is not tolerated and the mathematical operator must be explicitly included, resulting in $a \cdot b$. I have endeavored to make multiplication explicit in my own mathematical expression (even outside math programs), though it takes slightly more effort to write math this way. Yet it makes clear when multiplication is and is not involved.

Another ambiguity of mathematical notation that explicit use of the multiplication symbol resolves is whether the relationship between two symbols is multiplicative or functional. For instance, because of our common math tradition, we instinctively recognize $f(x)$ as functional notation though there is nothing to prevent this from being interpreted as $f \cdot x$. One of the side benefits of computers to human beings is that machines lack abstracting powers and require completely consistent notation. The human mind can put notational ambiguity in a larger interpretive context that the machine (ordinarily) cannot. In this case, the superior mental powers of the human mind lead to a simple-minded ambiguity and, in too many cases, can fool even human minds. It is generally better to be consistent when rigor is required than depend upon interpretive skill.

Units are thus able to be accounted for mathematically, and regarded as constants whose actual values must come from physical interpretation and cannot be computed or derived mathematically. This does not prevent us from combining them mathematically, and even defining them using predefined units. Systems of units, such as the metric (MKS, or the present refinement, the SI) or English (aka Imperial), reduce the number of independent units to only a few, such as length, time, mass, and charge. The metric system includes mass as a fundamental quantity and force is expressed in it and other fundamental quantities. In the English system, force is regarded as fundamental, leading to pounds-force (lbf) as a distinct quantity from pounds-mass (lbm) or slugs.

**Are There Divinely Given Units?**

As a side-note only tangentially related to electronics (yet bound to be interesting to some electronics engineers), the metric and English units of length are related exactly, historically. Over the last few hundred years, refinement of the definition of an inch has caused it to change by about 0.1% from its ancient source, rooted in the cubit, as derived from accurate length measurements of the Great Pyramid at Giza in Egypt. There are 25 pyramid inches per cubit. (One rather extensive work in print, *Great Pyramid Passages*, by John and Morton Edgar http://www.artisanpublications.com, has a section, "The Great Pyramid unit of measure", page 21 *ff*, that begins discussion of length units.) The cubit is defined as 1/10,000,000 the radius of the earth. The meter was originally defined as 1/10,000,000 the distance from the North Pole to the equator through a meridian that included Paris. That is, there are $10^4$ km per quarter the earth's circumference, if we lay the imperfect sphericity of the earth aside. Given that both the cubit, for which:

$$25 \text{ inches} \equiv 1 \text{ cubit}$$

and the meter are both grounded in the size of the earth, then the conversion from meters to cubits is exactly:

$$\frac{\pi}{2} \text{ cubit} \equiv 1 \text{ meter}$$

Because the inch traces back in this way to very ancient times, Britain was reticent to adopt the metric system in the 19th century. A couple of knighted astronomers, who were advisors to the government, believed that, because of the geometrically-coded message of human history in the Great Pyramid, the cubit and inch units were divinely given and chose to stay with them. (Britain's progeny in America have held onto them the longest.) The Great Pyramid is itself a fascinating mystery of the ancient world which has an intricate geometric interrelationship of constructs that only in the last two centuries have been coming to light. Whether the inch is a divinely-given unit or not, I leave that to you, dear reader, to decide. If so, the metric system, being tied exactly (historically) to it, would derivatively be divine in some sense. Lest we lose our sense of this article, let us return to the more mundane engineering application of units, or what give the appearance of units.



**Pseudo-Units**

Some mathematical constructs are sometimes treated as units but they are scale factors instead. The decibel (dB) is defined as:

$$x \text{ dB} \equiv 10 \cdot \log_{10}(x)$$

where, $x$ is expressed in dB.

Although "dB" is written after the number $x$ indicating how many decibels, dB is not really a unit. It cannot be expressed in the fundamental units of the SI system. It can only be applied to unitless quantities, or numbers: $x$ must be unitless. Sometimes "dB" is written in parentheses after the symbol for the quantity to show that "dB" is not a unit but that the quantity is scaled (logarithmically) in dB, as defined above. It can be used with quantities that have units, such as power, $p$, measured in units of watts, W, but only as ratios:

$$p(\text{dB}) = 10 \cdot \log\left(\frac{p}{P}\right)$$

where $p$ (dB) is power, expressed in decibels with respect to $P$, the reference amount of power that is 0 dB.

Logarithmic scaling is used to compress quantities that range over many decades into a more convenient scale, allowing us to think of amounts in powers of ten rather than the amounts themselves.

To be consistent with the above use of the dB scale to scale power, then voltage or current, *x*, both of which vary as the square with respect to power are expressed in a consistent logarithmic scale as:

$$x(dB) = 10 \cdot \log\left(\frac{x}{X}\right)^2 = 20 \cdot \log\left(\frac{x}{X}\right)$$

where *X* is the reference (0 dB) value of *x*.

In all cases, *x* and *X* must be in the same (or convertible) units so that their ratio is unitless.

Another quantity that is not really a unit either is the *radian* (rad). There are $2 \cdot \pi$ radians per revolution or cycle of a circle. In trigonometry, the measure of angle is defined based on the unit circle, a circle of radius one with center at the origin, $(x, y) = (0, 0)$ of the coordinate system. Then the definition of sine and cosine of angle $\theta$ is: $x = \cos\theta$ and $y = \sin\theta$. This is basic trigonometry and does not involve physical quantities directly. The distance along the circumference of the circle equal to the radius is subtended by an arc having an angle of 1 radian, by definition. Yet there is nothing requiring that a unit of angular measure be introduced. The angle subtended is a ratio of lengths, that along the circumference over the radius.

The motivation to call this ratio a unit arises because of other angular units. I know of three. Besides the grad, which I read is a European measure that divides the circle into 400 parts (an even 100 per quadrant) there are revolutions (or cycles) and degrees. The degree is another measure shrouded in ancient history, going back to the Chaldeans, with their base-60 mathematics and 360 day year. The year is, of course, associated with the earth going around the sun in the circle of its orbit, and it is not unexpected that the circle would be divided into 360 degrees of angle. This is more of a physically derived unit of angle - indeed, a geocentric measure – than purely mathematical. Consequently, the *degree* as an angular unit, despite how deeply entrenched it is in engineering and science, is an artificial and somewhat arbitrary measure. I have begun to disabuse myself of it, preferring instead the more natural measures of radian and revolution. In both DSP and control theory, for instance, these arise naturally, and to think in them simplifies conceptual understanding by eliminating an unnecessary construct, the degree.

In electrical engineering, frequency is expressed in inverse seconds, $s^{-1}$, though sometimes rad/s is used. However, in carrying out math on units, there is often nothing to cancel the radians. It is a pseudo-unit, given a name to distinguish it from degrees and revolutions. In this use, it functions well, but to apply it as a unit can lead to puzzlement and confusion. It is best not introduced as a unit into engineering calculations.

The relationship between radians and revolutions is simply that there are $2 \cdot \pi$ rad/rev. One occurrence of the need to resolve angular units (again, involving frequency) are in transfer functions in the *s*-domain. In normalized form, however, all transfer functions can be written as a ratio of frequencies. For instance:

$$G(s) = G_0 \cdot \left(\frac{\frac{s}{\omega_z} + 1}{\frac{s}{\omega_0}}\right)$$

has a zero at $\omega_z$ and a pole at the origin. The origin pole crosses the unity-gain (0 dB) axis at a frequency of $\omega_0$. Although $\omega_0$ is not a break frequency, it is the frequency to use in the ratio of the pole at the origin because at $s = \omega_0$ (or more properly, at $s = j\omega = j\omega_0$), $s/\omega_0 = 1$ just as the zero is at $s = \omega_z$. The *s*-dependent factor of $G(s)$ (the rational function in parentheses) is thereby reduced to one at dc ($s = 0\ s^{-1}$). Poles at the origin make the value of the transfer function magnitude infinite at 0 Hz, and it makes no sense in that case to talk of the static (dc) gain. It is not $G_0$; it is infinite. However, by setting the value of the “static gain,” $G_0$, as the value at $G(\omega_0)$, the transfer function in its *normalized form* still has a useful meaning when the *s*-dependent factor is equal to one.

With frequency ratios in use, the convention of using $\omega$ to represent frequency in $s^{-1}$ and $f$ in Hz (≡ rev/s) is somewhat of an artifice of convenience and not really a necessity. In any engineering equation, units must be reconciled, and in transfer functions for which quantities of frequency always occur in ratios, they can be either in $s^{-1}$ or Hz (or even degrees/second), as long as both of the quantities of the ratio have the same units and cancel.

**Units Conversion**

Finally, conversion from one unit to another for a given physical quantity can sometimes be perplexing. Most units differ from each other by a multiplier. Exceptions are temperature units Celsius, °C, and Kelvin, K, or Fahrenheit, °F, and Rankine, R; both pairs differ by a difference, not a scale factor. (This drives math programs such as MathCAD slightly crazy.) Perhaps this is another case where it might be best to settle for the natural units, K (and R), which in thermodynamics are seen to be units that fall out of the theory itself. (Display manufacturers could get behind this, for electronic displays of ambient temperature would require an additional digit, thereby increasing per-digit display sales by 50%.)

Returning to Hz and $s^{-1}$, in math programs, both reduce to the fundamental unit of time, s. The $2\cdot\pi$ is a scale factor, and, in MathCAD:

$$\text{Hz} \equiv s^{-1}/2\cdot\pi$$

Alternatively, defining $s^{-1}$ in Hz can cause trouble because Hz is not a fundamental unit.

Perhaps we are blessed to be electronics engineers instead of mechanical or chemical engineers, for we rarely have to deal with the plethora of units for pressure. It can be measured in inches or mm or µm of water or mercury, in psi, in torr or atmospheres or bars, or in pascals, the basic metric unit that is $N/m^2$, where N is a newton, the basic metric unit of force: $kg\cdot m/s^2$. Electricity and magnetism, having arrived relatively recently in human history, have no English units and are traditionally metric (though CGS ab-quantities are still around among some physicists). With the accelerated pace of conversion to metric (SI, that is), even circuit boards are now being laid out with a metric scale and as IC packages shrink, they shrink in conformance with metric pin pitches. In a few more years, the whole world will have a single system of measure. The next generation will not be (as) troubled with having to carry out tedious and conceptually unfruitful units conversions.